

Alternative to quantum adiabatic evolution using fixed-point quantum search

Avatar Tulsi

Department of Physics, IIT Bombay, Mumbai-400076, India*

A quantum system can be evolved from the ground state of an initial Hamiltonian to that of a final Hamiltonian by adiabatically changing the Hamiltonian with respect to time. The system remains in the ground-state of time-changing Hamiltonian provided the change is slow enough. More precisely, if g is the minimum energy gap between the ground state and other eigenstates of time-changing Hamiltonian then the evolution time must scale as the inverse square of g for a successful evolution. Childs et al. [1] proposed an alternative, where the system is kept in the ground state of a time-changing Hamiltonian by doing measurements at suitably small enough time intervals. Their scheme is successful only if the time scales as the inverse cube of g , and thus the time-scaling is inferior to the adiabatic evolution. Here, we propose another alternative which is essentially similar to the Childs' scheme but uses the concept of fixed-point quantum search (FPQS) algorithm [2, 3] to recover the inverse-square time-scaling behaviour of adiabatic evolution. Our algorithm uses selective transformations of the unknown ground states and phase-estimation algorithm (PEA) is the main tool to approximate such selective transformations. Though the approximation provided by PEA is not sufficient for our algorithm, we show that a combination of PEA and FPQS can significantly enhance the approximation making it sufficient for our algorithm. Thus we demonstrate interesting applications of fixed-point quantum search which achieves monotonic convergence towards the desired final state.

PACS numbers: 03.67.Ac

I. INTRODUCTION

Quantum adiabatic evolution starts with the ground state of an initial Hamiltonian H_0 and evolves it to the ground state of a final Hamiltonian H_1 . This is done by evolving under a time-dependent Hamiltonian

$$H_s = (1 - s)H_0 + sH_1, \quad s = t/T. \quad (1)$$

where t denotes time and T is the total evolution time. For a successful evolution, T should be large enough and the lower bound on it depends upon the eigenspectra of H_s . More precisely, let the eigenspectra of H_s be

$$H_s|E_{s,j}\rangle = E_{s,j}|E_{s,j}\rangle. \quad (2)$$

with $E_{s,0} \leq E_{s,1} \leq \dots \leq E_{s,N-1}$, i.e $|E_{s,0}\rangle$ denote the ground state of H_s and $E_{s,j}$ denote the j^{th} excited state of H_s . Here N is the dimension of the Hilbert space. We assume that H_s has a nondegenerate ground state for all s . We define the minimum energy gap g to be

$$g = \min_s g_s; \quad g_s = E_{s,1} - E_{s,0}, \quad (3)$$

and the quantity Γ as

$$\Gamma = \|H_0\| + \|H_1\|. \quad (4)$$

The quantum adiabatic theorem [4] states that the quantum adiabatic evolution is successful only if $T = O(\Gamma/g^2)$.

In quantum computing, Farhi et al. [5] suggested a potential application of quantum adiabatic evolution to solve combinatorial search problems, where the main goal is to minimize a cost function, defined according to the nature of problem. If we design the final Hamiltonian H_1 in such a way that its eigenvalues are the possible cost functions of the search problem then its ground state will correspond to the minimum cost function and hence getting the ground state will solve the problem. In such applications, g is the main factor to determine the performance of evolution as Γ is typically polynomial in problem size. A major open question in quantum computing is to estimate g and hence the running time T of evolution for combinatorial search problems.

Childs et.al. proposed an alternative algorithm [1] to quantum adiabatic evolution which uses a sequence of measurements instead of a continuous time evolution. Let $\Delta = 1/M$ be a small quantity. Then the ground state $|E_{s+\Delta,0}\rangle$ of $H_{s+\Delta}$ will be close to the ground state $|E_{s,0}\rangle$ of H_s . Hence if the system is in $|E_{s,0}\rangle$ then measuring $H_{s+\Delta}$ will yield $|E_{s+\Delta,0}\rangle$ with a high probability. If we begin with $|E_{0,0}\rangle$ and successively measure $H_{\Delta}, H_{2\Delta}, \dots, H_{(M-1)\Delta}, H_1$, then the final state will be the desired ground state $|E_{1,0}\rangle$ of H_1 with probability close to 1, provided Δ is small enough. The running time of Childs et al.'s algorithm is $T = O(\Gamma^2/g^3)$, which is inferior to that of adiabatic evolution by a factor of Γ/g .

In this paper, we show that if we use fixed-point quantum search algorithm [2, 3] instead of measurements in the Childs et.al's algorithm then its running time has almost same scaling as that of quantum adiabatic evolution. The paper is organized as follows. In next section, we motivate the readers by presenting the basic

*Electronic address: tulsi9@gmail.com

idea of our algorithm. For our algorithm, we need selective transformations of the unknown ground states and we approximate such transformations using phase estimation algorithm (PEA). But PEA does not provide an approximation sufficient for the success of our algorithm. In Sec. III, we show that a combination of PEA and FPQS can significantly enhance the approximation making it sufficient for our algorithm. We conclude in section IV.

II. BASIC IDEA

Here we present the basic idea of our algorithm which is basically a variant of Childs et.al's algorithm where we use the fixed-point quantum search algorithm (FPQS) to enhance its performance. First, we discuss these two algorithms and then we present our algorithm.

A. Childs et.al's algorithm

Childs et.al's algorithm uses the fact that the ground state of $H_{s+\Delta}$ is close to the ground state of H_s for small Δ . To make it more precise, we use the perturbation theory. Using Eq. (1), we get

$$H_{s+\Delta} = H_s + \Delta(H_1 - H_0). \quad (5)$$

For small Δ , we can treat the second term of R.H.S. as a small perturbation. Using perturbation theory and ignoring $O(\Delta^3)$ terms, we find that

$$|\langle E_{s+\Delta} | E_s \rangle|^2 = 1 - \Delta^2 \sum_{j \neq 0} \frac{|\langle E_{s,0} | (H_0 - H_1) | E_{s,j} \rangle|^2}{(E_{s,0} - E_{s,j})^2} + O(\Delta^3). \quad (6)$$

Since $|\langle E_{s,0} | (H_0 - H_1) | E_{s,j} \rangle| \leq \Gamma$ and $E_{s,j} - E_{s,0} \geq g$ by definition, we get

$$|\langle E_{s+\Delta} | E_s \rangle|^2 \geq 1 - \frac{\Delta^2 \Gamma^2}{g^2}. \quad (7)$$

Hence measuring $H_{s+\Delta}$, when the system is in the state $|E_{s,0}\rangle$, will give the state $|E_{s+\Delta,0}\rangle$ with a minimum probability of $1 - (\Delta\Gamma/g)^2$.

In Childs et.al's algorithm, we start with the state $|E_{0,0}\rangle$. We choose $\Delta = 1/M$ and then we do M successive measurements of the Hamiltonians $H_\Delta, H_{2\Delta}, \dots, H_{(M-1)\Delta}, H_1$. At each measurement, we get the ground state of measured Hamiltonian with a minimum probability of $1 - (\Gamma/Mg)^2$. Hence the probability to get the ground state of the final Hamiltonian H_1 is given by $P \geq (1 - (\Gamma/Mg)^2)^M$, which is close to 1 provided $M \gg \Gamma^2/g^2$. Thus the number of measurements required by Childs et. al's algorithm is

$$M = O\left(\frac{\Gamma^2}{g^2}\right). \quad (8)$$

To measure the Hamiltonian H_s , Childs et al's algorithm uses a scheme similar to the phase-estimation algorithm (PEA). As shown by them, the time required to do a single measurement τ scales as $O(1/g)$. The total running time of Childs et al's algorithm is the product of required number of measurements M and the time needed per measurement $\tau = O(1/g)$ which comes out to be

$$M\tau = O\left(\frac{\Gamma^2}{g^3}\right). \quad (9)$$

B. Fixed-point quantum search

The fixed-point quantum search algorithm (FPQS) [2, 3] was originally discovered to supplement the standard quantum search algorithm [7]. In the quantum search algorithm, the main goal is to drive the quantum system from an initial state $|\alpha\rangle$ to an unknown final state $|\beta\rangle$ by using oracular access to these states. The oracular access to a state implies the ability to perform a selective operation on that state. The standard quantum search algorithm, though proved to be optimal for $|\langle \alpha | \beta \rangle| \ll 1$, does not work so efficiently when $|\langle \alpha | \beta \rangle| \approx 1$ and we need FPQS for optimal performance in latter case.

In FPQS, we make use of two operators R_α and R_β , which are the selective $\frac{\pi}{3}$ -phase rotations of states $|\alpha\rangle$ and $|\beta\rangle$ respectively, and their inverses. Mathematically, we have

$$R_\alpha = 1 - (1 - e^{i\pi/3})|\alpha\rangle\langle\alpha|, \quad (10)$$

$$R_\beta = 1 - (1 - e^{i\pi/3})|\beta\rangle\langle\beta|. \quad (11)$$

FPQS is based on the result that if $|\langle \beta | \alpha \rangle|^2 = 1 - \epsilon$ then $|\langle \beta | R_\alpha R_\beta | \alpha \rangle|^2 = 1 - \epsilon^3$, i.e.,

$$|\langle \beta | \alpha \rangle|^2 = 1 - \epsilon \implies |\langle \beta | R_\alpha R_\beta | \alpha \rangle|^2 = 1 - \epsilon^3. \quad (12)$$

Since the above equation holds true for any two quantum states $|\alpha\rangle, |\beta\rangle$, it also holds true for $|\alpha_1\rangle, |\beta\rangle$, where $|\alpha_1\rangle = R_\alpha R_\beta |\alpha\rangle$. Then above equation gives us

$$|\langle \beta | \alpha_1 \rangle|^2 = 1 - \epsilon^3 \implies |\langle \beta | R_{\alpha_1} R_\beta | \alpha_1 \rangle|^2 = 1 - \epsilon^9. \quad (13)$$

Let $V_1 = R_\alpha R_\beta$. As $|\alpha_1\rangle = V_1 |\alpha\rangle$, we have

$$\begin{aligned} R_{\alpha_1} &= 1 - (1 - e^{i\pi/3})|\alpha_1\rangle\langle\alpha_1| \\ &= 1 - (1 - e^{i\pi/3})V_1 |\alpha\rangle\langle\alpha| V_1^\dagger \\ &= V_1 [1 - (1 - e^{i\pi/3})|\alpha\rangle\langle\alpha|] V_1^\dagger \\ &= V_1 R_\alpha V_1^\dagger. \end{aligned} \quad (14)$$

Similarly, we can recursively define the following relations

$$\begin{aligned} |\alpha_{n+1}\rangle &= R_{\alpha_n} R_\beta |\alpha_n\rangle \equiv V_{n+1} |\alpha\rangle, \quad |\alpha_0\rangle = |\alpha\rangle, \\ |\langle \beta | \alpha_n \rangle|^2 &= 1 - \epsilon^{3^n}, \\ R_{\alpha_n} &= V_n R_\alpha V_n^\dagger, \\ V_{n+1} &= V_n R_\alpha V_n^\dagger R_\beta V_n, \quad V_0 = 1. \end{aligned} \quad (15)$$

Let \mathcal{R} denote the set $\{R_a, R_b, R_a^\dagger, R_b^\dagger\}$. We see that each element of \mathcal{R} requires an oracle for implementation. The time complexity of FPQS algorithm is measured in terms of q_n , the number of elements of \mathcal{R} used by the algorithm to obtain the state $|\alpha_n\rangle$. Eq. (15) gives us the following

$$\begin{aligned} q_{n+1} = 3q_n + 2 &\implies q_n = 3^n - 1, \\ |\langle\beta|\alpha_n\rangle|^2 &= 1 - \epsilon^{q_n+1}. \end{aligned} \quad (16)$$

So we see that with each level of recursion, the failure probability, i.e., the probability of not getting $|\beta\rangle$ state upon measuring $|\alpha_n\rangle$ state decreases monotonically and hence we achieve a monotonic convergence towards the desired final state. From now onwards, we will use n-FPQS to denote FPQS algorithm of n^{th} level recursion.

FPQS is proved to be the optimal algorithm to achieve monotonic convergence [8]. The above-mentioned recursive scheme is optimal as long as the desired failure probability is of the form ϵ^{3^n} . There is another version of FPQS [3] which achieves the optimal performance when the desired failure probability is of the form ϵ^{2n+1} for any integer n . For the purpose of our algorithm, this version is not so relevant.

C. NEW ALGORITHM

The basic idea of our algorithm is to use the monotonic convergence of FPQS to enhance the performance of Childs et al's algorithm. Using Eq. (7), we find that $|\langle E_{s+\Delta}|E_s\rangle|^2 \geq 1 - (\Gamma/Mg)^2$, where $M = 1/\Delta \gg 1$. Let $|\alpha\rangle = |E_s\rangle$ and $|\beta\rangle = |E_{s+\Delta}\rangle$ then

$$\epsilon = 1 - |\langle\beta|\alpha\rangle|^2 \leq (\Gamma/Mg)^2. \quad (17)$$

Now, we can use FPQS to converge towards $|E_{s+\Delta}\rangle$ state by using selective phase rotations of $|E_s\rangle$ and $|E_{s+\Delta}\rangle$ states. As we show in next section that selective phase rotations of $|E_s\rangle$ and $|E_{s+\Delta}\rangle$ states can always be approximated to a desired accuracy using PEA. For simplicity, right now we assume a perfect implementation of these selective transformations. Using Eqs. (16) and (17), we can write this fact as

$$|\langle E_{s+\Delta}|\tilde{R}^{q_n}|E_s\rangle|^2 \geq 1 - (\Gamma/Mg)^{q_n}, \quad q_n = 3^n - 1, \quad (18)$$

where \tilde{R} represents any element of the set \mathcal{R} of selective transformations, i.e., $\mathcal{R} = \{R_{E(s,0)}, R_{E(s+\Delta,0)}, R_{E(s,0)}^\dagger, R_{E(s+\Delta,0)}^\dagger\}$. Thus, we monotonically converge towards the ground state of $H_{s+\Delta}$ by using FPQS.

In our algorithm, we start with the ground state $|E_{0,0}\rangle$ of H_0 , which is easy to prepare. Then we use n-FPQS to converge towards $|E_{\Delta,0}\rangle$, ground state of H_Δ , and then do a projective measurement on $|E_{\Delta,0}\rangle$ to get $|E_{\Delta,0}\rangle$ with a high probability. Then we repeat the same process, i.e., we use n-FPQS to converge towards $|E_{2\Delta,0}\rangle$ and then measure to get $|E_{2\Delta,0}\rangle$ with a high probability. We repeat the similar process to get the ground states of

$H_{3\Delta}, H_{4\Delta}, \dots, H_{(M-1)\Delta}, H_1$. There are total of M measurements. At each measurement, the probability of getting the desired ground state is at least $1 - (\Gamma/Mg)^{2(q_n+1)}$ according to Eq. (16) and hence the success probability of getting the final ground state $|E_{1,0}\rangle$ is

$$P \geq (1 - (\Gamma/Mg)^{2(q_n+1)})^M = 1 - \frac{1}{M^{2q_n+1}} \left(\frac{\Gamma}{g}\right)^{2q_n+2}. \quad (19)$$

For large q_n , this probability is close to 1 if

$$M = O\left((\Gamma/g)^{1+(2q_n)^{-1}}\right) \quad (20)$$

Here we can understand the main reason behind improvement over Childs et al's algorithm. The minimum number of measurements M , required by Childs et. al's algorithm is $(\Gamma/g)^2$ (see Eq. 8) whereas our algorithm requires only $(\Gamma/g)^{1+(2q_n)^{-1}}$ measurements. Before each measurement, we are using q_n selective transformations to converge towards desired ground state. Hence the total number of selective transformations used by our algorithm is

$$O\left(q \left(\frac{\Gamma}{g}\right)^{1+\frac{1}{2q}}\right) \quad (21)$$

where we have dropped the subscript n for convenience. The quantity within the bracket is minimum for $q = \ln(\Gamma/g)/2$, putting which we find that the minimum number of selective transformations Q required by our algorithm is

$$Q = O\left(\frac{\Gamma}{g} \ln \frac{\Gamma}{g}\right). \quad (22)$$

Next, we discuss the implementation of selective transformations \tilde{R} using PEA and we find the time required to implement one selective transformation.

III. APPROXIMATING SELECTIVE TRANSFORMATIONS

Here we discuss the implementation of $R_{E(s,0)}$, selective phase rotation of the ground state $|E_{s,0}\rangle$ of H_s . Let the state of our quantum system be

$$|\psi\rangle = \sum_j a_j |E_{s,j}\rangle, \quad (23)$$

we refer to this system as *main* system. To implement $R_{E(s,0)}$, we need to transform it to

$$R_s|\psi\rangle = \sum_j a_j e^{i\omega\delta_{j,0}} |E_{s,j}\rangle \quad (24)$$

where $\delta_{j,0}$ is the Kronecker's delta symbol. It can be done easily if we have the knowledge of the eigenspectra of H_s but in the absence of any knowledge, the best

way to approximate this transformation is to use PEA to maximally distinguish among the eigenstates of H_s . In PEA, we attach an *ancilla* quantum system of l qubits having 2^l -dimensional Hilbert space with the basis states $|z\rangle$, $z \in \{0, 1, \dots, 2^l - 1\}$, encoded by l -bit binary strings. The initial state of ancilla system is the equal superposition $|e\rangle = 2^{-l/2} \sum_z |z\rangle$ of all these basis states. Thus the initial state is

$$|\psi_0\rangle = |\psi\rangle_1 |e\rangle_2 = \frac{1}{2^{l/2}} \sum_{j,z} a_j |E_{s,j}\rangle_1 |z\rangle_2, \quad (25)$$

where the subscripts 1 and 2 refer to the main and ancilla quantum system respectively.

A. Scheme of approximation

Let us first discuss our general scheme to approximate selective transformations. Let us define a marked set $\{A\}$ as a subset of the set $\{z\}$ of 2^l basis states of the ancilla system. Let us define two mutually orthogonal subspaces of ancilla Hilbert space as a marked subspace, spanned by the elements of $\{A\}$, and an unmarked subspace, spanned by the elements of $\{A_\perp\} = \{z\} - \{A\}$. Let any state completely within the marked subspace be denoted as a marked state $|m\rangle = \sum_{z \in A} b_z |z\rangle$ while a state completely within the unmarked subspace be denoted as an unmarked state $|m_\perp\rangle = \sum_{z \notin A} b_z |z\rangle$. Let us define a unitary transformation \mathcal{B} as

$$\mathcal{B} = \sum_j |E_{s,j}\rangle \langle E_{s,j}|_1 \otimes (\mathcal{B}_j)_2. \quad (26)$$

Thus \mathcal{B} applies a unitary operator \mathcal{B}_j on the ancilla system if and only if the main system is in the state $|E_{s,j}\rangle$. Suppose \mathcal{B} satisfies

$$\mathcal{B}_j |e\rangle = \delta_{j,0} |m\rangle + (1 - \delta_{j,0}) |m_\perp\rangle. \quad (27)$$

Note that $|m_\perp\rangle$ can be mutually different states for different j 's, all we require is that it should be completely within the unmarked subspace for all $j \neq 0$. Now, we have

$$|\psi_1\rangle = \mathcal{B} |\psi_0\rangle = a_0 |E_{s,0}\rangle |m\rangle + \sum_{j \neq 0} a_j |E_{s,j}\rangle |m_\perp\rangle \quad (28)$$

where we have omitted the subscripts 1 and 2 for convenience. If we know the elements of marked set A then it is always possible to perform a selective phase rotation by any angle ω of $|m\rangle$ state by selectively rotating the phases of elements of $\{A\}$. Thus we get the state

$$|\psi_2\rangle = a_0 |E_{s,0}\rangle e^{i\omega} |m\rangle + \sum_{j \neq 0} a_j |E_{s,j}\rangle |m_\perp\rangle. \quad (29)$$

Then we apply \mathcal{B}^\dagger operator on the ancilla system to drive it back to its initial state $|e\rangle$ to get

$$|\psi_3\rangle = \left(\sum_j a_j e^{i\omega \delta_{j,0}} |E_{s,j}\rangle \right) |e\rangle. \quad (30)$$

The projection of above state on the main system space is exactly the state we are looking for. Thus we can implement a perfect selective phase rotation on the main system provided we can implement a transformation \mathcal{B} with the property given by Eq. (27). In general, implementing such a transformation is difficult in lack of knowledge of eigenspectra of H_s . So, we relax the condition of perfect transformation and aim only to approximate the selective transformation to a desired accuracy.

Instead of Eq. (27), we now look for a \mathcal{B} which satisfies the following relation

$$\begin{aligned} \mathcal{B}_j |e\rangle &= \gamma_j |m\rangle + \sqrt{1 - \gamma_j^2} |m_\perp\rangle, \\ \gamma_j &= \delta_{j,0} - \eta_j, \quad |\eta_j| \leq \eta \ll 1. \end{aligned} \quad (31)$$

As noted earlier, $|m\rangle$ and $|m_\perp\rangle$ can be different for different j 's, we just need them to be completely within the marked and unmarked subspace respectively. For simplicity, we choose phases of these states such that γ_j and so η_j is a real quantity. Ideally, η should be zero to satisfy Eq. (27) but in general, its hard to achieve. So if the main system is in the ground (excited) state, \mathcal{B} transforms the ancilla system from its initial state $|e\rangle$ to a state close to marked (unmarked) state. In this case, we find the corresponding $|\psi_3\rangle$ state to be

$$|\psi_3\rangle = \sum_j a_j |E_{s,j}\rangle \{ \mathcal{B}_j^\dagger (\gamma_j e^{i\omega} |m\rangle + \sqrt{1 - \gamma_j^2} |m_\perp\rangle) \}. \quad (32)$$

As γ_j are real, we have $\langle e | \mathcal{B}_j^\dagger | m \rangle = \langle m | \mathcal{B}_j | e \rangle = \gamma_j$ and $\langle e | \mathcal{B}_j^\dagger | m_\perp \rangle = \langle m_\perp | \mathcal{B}_j | e \rangle = \sqrt{1 - \gamma_j^2}$. So, above state can be written as

$$|\psi_3\rangle = \sum_j a_j |E_{s,j}\rangle (\mu_j |e\rangle + \sqrt{1 - \mu_j^2} |\perp\rangle), \quad (33)$$

where $|\perp\rangle$ is a normalised state orthogonal to $|e\rangle$ (it can be different for different j 's) and

$$\begin{aligned} \mu_j &= 1 + \gamma_j^2 (e^{i\omega} - 1) \\ &= 1 + (\delta_{j,0} - \eta_j)^2 (e^{i\omega} - 1) \\ &= e^{i\omega \delta_{j,0}} + (1 - e^{i\omega}) (2\eta_j \delta_{j,0} - \eta_j^2). \end{aligned} \quad (34)$$

We find that

$$\begin{aligned} |\mu_0| &= 1 - O(\eta) \quad \angle \mu_0 = \omega + O(\eta) \\ |\mu_{j \neq 0}| &= 1 - O(\eta^2) \quad \angle \mu_{j \neq 0} = O(\eta^2) \end{aligned} \quad (35)$$

With this, it is easy to see that after getting $|\psi_3\rangle$ state, if we measure the ancilla system, we will get the $|e\rangle$ state with $1 - O(\eta)$ probability and the main system will collapse to the following state

$$\begin{aligned} \frac{1}{1 - O(\eta)} \sum_j a_j \mu_j |E_{s,j}\rangle &= \sum_j a_j e^{i\omega \delta_{j,0}} |E_{s,j}\rangle + |\eta\rangle \\ &= R_s |\psi\rangle + |\eta\rangle \end{aligned} \quad (36)$$

where $|\eta\rangle$ is a vector of length $O(\eta)$ and we have made use of Eqs. (24) and (34). So we can approximate the selective transformation to the accuracy of η with a success probability $1 - O(\eta)$. Next, we show that using FPQS algorithm, we can effectively reduce η to improve our approximation.

B. Improving approximation

First, let us use FPQS to converge towards marked state $|m\rangle$. We define

$$|\alpha(j)\rangle = \mathcal{B}_j|e\rangle, \quad |\beta\rangle = |m\rangle \quad (37)$$

in FPQS algorithm discussed earlier. The necessary selective transformations for FPQS are

$$R_\beta = R_m, \quad R_{\alpha(j)} = \mathcal{B}_j R_e \mathcal{B}_j^\dagger. \quad (38)$$

They can be implemented accurately using the transformations \mathcal{B} and \mathcal{B}^\dagger and our knowledge of the marked set A and the state $|e\rangle$. Eq. (16) says that as long as $q = 3^n - 1$ for any integer n , using q applications of above selective transformations, we can construct an operator $\mathcal{B}(q)$ with the property

$$\begin{aligned} \mathcal{B}(q) &= \sum_j |E_{s,j}\rangle \langle E_{s,j}| \otimes \mathcal{B}_j(q) \\ \mathcal{B}_j(q)|e\rangle &= \sqrt{1 - (1 - \gamma_j^2)^{q+1}} |m\rangle + (1 - \gamma_j^2)^{\frac{q+1}{2}} |m_\perp\rangle \end{aligned} \quad (39)$$

Using the facts that $\gamma_j = \delta_{j,0} - \eta_j$ and $\eta_j \ll 1$, we find the following relations upto leading order approximation in η_j ,

$$\begin{aligned} \mathcal{B}_0(q)|e\rangle &= \left(1 - \frac{(2\eta_0)^{q+1}}{2}\right) |m\rangle + (2\eta_0)^{\frac{q+1}{2}} |m_\perp\rangle \\ \mathcal{B}_{j \neq 0}(q)|e\rangle &= \sqrt{q+1}\eta_j |m\rangle + \left(1 - \frac{q+1}{2}\eta_j^2\right) |m_\perp\rangle. \end{aligned} \quad (40)$$

Note that the second approximation holds only if

$$q+1 \ll 2/\eta_{j \neq 0}^2. \quad (41)$$

As $\mathcal{B}(q)$ uses q applications of the selective transformations given by Eq. (38), it uses a maximum of $2q$ applications of either \mathcal{B} or \mathcal{B}^\dagger .

Now, we find from Eq. (40) that while $\eta_0(q) = O(\eta_0^{q+1})$ decreases rapidly with increasing q , the same is not true for $\eta_{j \neq 0}(q) = \sqrt{q+1}\eta_{j \neq 0}$ which increases with increasing q . As we have to reduce $\eta_j(q)$ for all j , we again use FPQS but this time to converge towards $|m_\perp\rangle$. Thus we define the states

$$|\alpha(j, q)\rangle = \mathcal{B}_j(q)|e\rangle, \quad |\beta\rangle = |m_\perp\rangle \quad (42)$$

in FPQS algorithm. Then the necessary selective transformations for FPQS are

$$R_\beta = R_{m_\perp}, \quad R_{\alpha(j, q)} = \mathcal{B}_j(q) R_e \mathcal{B}_j(q)^\dagger \quad (43)$$

which can be implemented accurately using the transformations $\{\mathcal{B}(q), \mathcal{B}(q)^\dagger\}$ and our knowledge of the marked set A and the state $|e\rangle$. Using Eq. (16) we find that as long as $q' = 3^n - 1$ for any integer n , using q' applications of above selective transformations, we can construct an operator $\mathcal{B}(q, q')$ with the property

$$\begin{aligned} \mathcal{B}(q, q') &= \sum_j |E_{s,j}\rangle \langle E_{s,j}| \otimes \mathcal{B}_j(q, q') \\ \langle m | \mathcal{B}_0(q, q') | e \rangle &= 1 - \frac{q' + 1}{2} (2\eta_0)^{q+1} \\ \langle m | \mathcal{B}_{j \neq 0}(q, q') | e \rangle &= (\sqrt{q+1}\eta_j)^{q'+1} \end{aligned} \quad (44)$$

So we find that

$$\begin{aligned} \eta_0(q, q') &= \frac{q' + 1}{2} (2\eta_0)^{q+1}, \\ \eta_{j \neq 0}(q, q') &= (\sqrt{q+1}\eta_{j \neq 0})^{q'+1}. \end{aligned} \quad (45)$$

By a suitable choice of q and q' , we can make sure that $\eta_j(q, q')$ decreases for all j . Note that $\mathcal{B}(q, q')$ uses q' applications of the selective transformations given by Eq. (43) hence it uses a maximum of $2q'$ applications of $\{\mathcal{B}(q), \mathcal{B}(q)^\dagger\}$, which uses $2q$ applications of $\{\mathcal{B}, \mathcal{B}^\dagger\}$. Thus we need a maximum of $4qq'$ applications of either \mathcal{B} or \mathcal{B}^\dagger to construct $\mathcal{B}(q, q')$. For the special case of $q = q'$, $\mathcal{B}(q, q)$ uses $O(q^2)$ applications of \mathcal{B} to get

$$\eta(q, q) = O((q\eta)^{q+1}). \quad (46)$$

Next, we illustrate this general scheme of approximating selective transformation by specific case of phase estimation algorithm.

C. Phase-estimation algorithm

Phase estimation algorithm can be used to approximately distinguish among the unknown eigenstates of a given Hamiltonian H_s . Let U_s be the unitary time-evolution operator corresponding to $2\pi H_s$, i.e. $U_s = e^{-i2\pi H_s t}$. As $\|H_s\| = O(\Gamma)$ by Eq. (4), we choose

$$t \leq (2\pi\|H_s\|)^{-1} = O(1/\Gamma) \quad (47)$$

to ensure one-to-one correspondence between the eigen-spectra of H_s and U_s . Hence the eigenstates of U_s are $|E_{s,j}\rangle$ with the eigenvalues $e^{-i2\pi E_{s,j}t}$. The crucial component of PEA is the Quantum Fourier Transform \mathcal{F} . In a 2^l -dimensional Hilbert space with the basis states $|0\rangle, |1\rangle, \dots, |2^l - 1\rangle$, \mathcal{F} can be defined by its action on each basis state, i.e.

$$\mathcal{F}|z\rangle = \frac{1}{2^{l/2}} \sum_{k=0}^{2^l-1} \exp(2\pi i k z / 2^l) |k\rangle. \quad (48)$$

PEA uses controlled- U_s^z transformation, denoted by C_U here, and \mathcal{F} on the initial state $|\psi_1\rangle$ to get

$$\begin{aligned}
& \frac{1}{2^{l/2}} \sum_{j,z} a_j |E_{s,j}\rangle |z\rangle \\
& \xrightarrow{C_U} \frac{1}{2^{l/2}} \sum_{j,z} a_j U^z |E_{s,j}\rangle |z\rangle \\
& \equiv \frac{1}{2^{l/2}} \sum_{j,z} a_j |E_{s,j}\rangle (e^{-i2\pi z E_{s,j} t} |z\rangle) \\
& \xrightarrow{\mathcal{F}} \sum_j a_j |E_{s,j}\rangle |\phi_{s,j}\rangle, \tag{49}
\end{aligned}$$

where, the state $|\phi_{s,j}\rangle$ can be written using Eq. (48) as

$$|\phi_{s,j}\rangle = \frac{1}{2^l} \sum_{k,z=0}^{2^l-1} \exp[2\pi i z 2^{-l} (k - 2^l E_{s,j} t)] |k\rangle. \tag{50}$$

This is the standard output state of PEA and has been analyzed well in literature (for example, see Sec. 5.2.1 of [6]). To get an idea, note that we have

$$\langle k | \phi_{s,j} \rangle = \frac{1}{2^l} \sum_{z=0}^{2^l-1} (\exp[2\pi i z 2^{-l} (k - 2^l E_{s,j} t)])^z, \tag{51}$$

which is the sum of a geometric series, so

$$\langle k | \phi_{s,j} \rangle = \frac{1}{2^l} \left(\frac{1 - \exp[2\pi i (k - 2^l E_{s,j} t)]}{1 - \exp[2\pi i 2^{-l} (k - 2^l E_{s,j} t)]} \right) \tag{52}$$

or

$$|\langle k | \phi_{s,j} \rangle| = \frac{\sin[\pi(k - 2^l E_{s,j} t)]}{2^l \sin[\pi 2^{-l} (k - 2^l E_{s,j} t)]}. \tag{53}$$

If $2^l E_{s,j} t$ is very close to an integer $\mathcal{N}_{s,j}$, i.e. $|\mathcal{N}_{s,j} - 2^l E_{s,j} t| \ll 1$ then using $\sin x \approx x$ for $x \ll 1$, we find that $|\langle \mathcal{N}_{s,j} | \phi_{s,j} \rangle| \approx 1$. Since $|\phi_{s,j}\rangle$ is a normalised state, measuring $|\phi_{s,j}\rangle$ will give us the state $|\mathcal{N}_{s,j}\rangle$ with high probability. Note that the state $|\mathcal{N}_{s,j}\rangle$ encodes the eigenvalue $E_{s,j}$ as $\mathcal{N}_{s,j} \approx 2^l E_{s,j} t$.

However, in general, $2^l E_{s,j} t$ may not be very close to an integer but even then, measurement of $|\phi_{s,j}\rangle$ will output an integer very close to $2^l E_{s,j} t$. Precisely, let $\mathcal{N}_{s,j} = \lfloor 2^l E_{s,j} t \rfloor$ be the highest integer less than $2^l E_{s,j} t$. Let c be another positive integer. As shown in Sec. 5.2.1 of [6], the probability of getting a k such that $|k - \mathcal{N}_{s,j}| \bmod 2^l \leq c$ after measuring $|\phi_{s,j}\rangle$ is given by

$$p(|k - \mathcal{N}_{s,j}| \leq c) = \sum_{|k - \mathcal{N}_{s,j}| \leq c} |\langle k | \phi_{s,j} \rangle|^2 \geq 1 - \frac{1}{2(c-1)}, \tag{54}$$

where we have removed the $\bmod 2^l$ notation for simplicity. Thus, with a probability of $1 - \frac{1}{2(c-1)}$, measuring $|\phi_{s,j}\rangle$ gives an integer k which differs with $2^l E_{s,j} t$ by at most c .

Hence, we get a good approximation for $E_{s,j}$ if we choose l to be large enough.

Let \mathcal{P} denote the transformation corresponding to PEA. It is obvious that \mathcal{P} is of the form $\sum_j |E_{s,j}\rangle \langle E_{s,j}| \otimes \mathcal{P}_j$ where $\mathcal{P}_j |e\rangle = |\phi_{s,j}\rangle$. Now, to approximate the selective transformation, we want $|\phi_{s,0}\rangle$ to have very small components in $|\phi_{s,j \neq 0}\rangle$ states. We note that for large l ,

$$\mathcal{N}_{s,j \neq 0} - \mathcal{N}_{s,0} \geq 2^l g t, \tag{55}$$

where g is the minimum energy gap defined by Eq. (3). We define our marked set to be set of all integers m satisfying

$$\mathcal{N}_{s,0} \leq m \leq \mathcal{N}_{s,0} + \mathcal{C}, \tag{56}$$

where \mathcal{C} is any integer satisfying $\mathcal{C} \leq 2^{l-1} g t$. By virtue of Eq. (55), we also get

$$\mathcal{N}_{s,j \neq 0} - m \geq 2^{l-1} g t. \tag{57}$$

Then Eq. (54) implies that

$$|\langle \phi_{s,0} | m \rangle|^2 \geq 1 - \frac{1}{2^l g t}, \quad |\langle \phi_{s,j \neq 0} | m \rangle|^2 \leq \frac{1}{2^l g t}. \tag{58}$$

Thus for \mathcal{P} ,

$$\eta_0 \leq \frac{1}{2^{l+1} g t}, \quad \eta_{j \neq 0} \leq \frac{1}{\sqrt{2^l g t}}, \quad \eta = \frac{1}{\sqrt{2^l g t}}. \tag{59}$$

Here t cannot be made arbitrarily large due to Eq. (47). So, only way to reduce η is to increase l and as Eq. (49) implies that \mathcal{P} requires 2^l applications of U_s , we improve the approximation at the expense of algorithm's running time. As we show later, a better way to improve the approximation is to use FPQS as discussed in the last section. We recall that to implement the selective transformation, we need the knowledge of marked set. Next, we discuss this aspect.

D. Finding the marked set

To find the marked set, we need to know $\mathcal{N}_{s,0}$ to a desired accuracy. One way to do this is to measure the output state of PEA given by Eq. (49). As for sufficiently large l , η is close to 0, $|\phi_{s,0}\rangle$ will be almost orthogonal to $|\phi_{s,j \neq 0}\rangle$ and it will also give a good estimate of $\mathcal{N}_{s,0}$. But the probability of getting this state is a_0^2 and after getting this as measurement output, the main system will collapse from its actual state $|\psi\rangle = \sum_j a_j |E_{s,j}\rangle$ to the ground state $|E_{s,0}\rangle$ of H_s . So this process disturbs the state of main system and hence it interferes with the working of our algorithm.

To overcome this limitation, we recall that our algorithm consists of M steps where in each step the initial state of system is $|E_{r\Delta,0}\rangle$ (r is an integer) and we evolve it to $|E_{(r+1)\Delta,0}\rangle$ by using selective transformations of these states. During evolution, the quantum system may have

arbitrary amplitudes in these states, but at the beginning, the system is guaranteed to be in $|E_{r\Delta,0}\rangle$ state. The idea is that before we start evolution, we use PEA to get a good estimate of $\mathcal{N}_{r\Delta,0}$. This won't disturb the system as its original state $|\psi\rangle$ is also the desired post-measurement state $|E_{r\Delta,0}\rangle$. Then we use perturbation theory to get an estimate for $\mathcal{N}_{(r+1)\Delta,0}$. Precisely, using Eqs. (4,5) and perturbation theory, we find that

$$|E_{s+\Delta,0} - E_{s,0}| = \Delta |\langle E_{s,0} | H_0 - H_1 | E_{s,0} \rangle| \leq \Delta \Gamma \quad (60)$$

which is true for any s including $s = r\Delta$. So we get

$$|\mathcal{N}_{(r+1)\Delta,0} - \mathcal{N}_{r\Delta,0}| \leq 2^l t \Delta \Gamma. \quad (61)$$

To estimate $\mathcal{N}_{r\Delta,0}$, we use PEA with the initial state $|E_{r\Delta,0}\rangle|e\rangle$ to get the state $|E_{r\Delta,0}\rangle|\phi_{r\Delta,0}\rangle$. Putting $c = 2$ in Eq. (54), we get where $|\phi_{r\Delta,0}\rangle$ satisfies the property

$$\sum_{|k - \mathcal{N}_{r\Delta,0}| \leq 2} |\langle \phi_{r\Delta,0} | k \rangle|^2 \geq \frac{1}{2}. \quad (62)$$

By repeating PEA a constant number of times, we can easily find $\mathcal{N}_{r\Delta,0}$ to an accuracy of ± 2 . By choosing

$$\Delta \leq g/2\Gamma \quad (63)$$

in Eq. (61), we can also know $\mathcal{N}_{(r+1)\Delta,0}$ to an accuracy of $2^{l-1}tg$ and thus we can know the marked set as defined by Eq. (56). From Eq. (20), we find that $\Delta = 1/M = (g/\Gamma)^{1+\frac{1}{2q_n}}$, thus above condition is always satisfied by our choice of $q_n = O(\ln(\Gamma/g))$. We also note from Eq. (55) that l should be chosen such that

$$2^l gt \gg 1 \quad (64)$$

in order to ensure that \mathcal{N}_{s0} is well-separated from $\mathcal{N}_{sj \neq 0}$.

IV. RUNNING TIME OF ALGORITHM

The total number of selective transformations used by our algorithm is given by Eq. (22) as

$$Q = O(x \ln x); \quad x = \Gamma/g. \quad (65)$$

Also, Eq. (36) implies that each selective transformation can be approximated upto an accuracy of $O(\eta)$ with a success probability of $1 - O(\eta)$. Hence our approximation will be good enough for the entire running of our algorithm as long as

$$\eta x \ln x \ll 1 \quad (66)$$

is satisfied. From Eq. (59), we have $\eta = (2^l gt)^{-1/2}$, so above condition is satisfied only if $2^l = O(x^2 \ln^2 x / gt)$. As approximation of each selective transformation using PEA requires 2^l application of U_s , we find that total applications of U_s , used by our algorithm is $O(2^l x \ln x) =$

$O(x^3 \ln^3 x / gt)$ which is greater than $O(\Gamma^3 / g^4)$. This scaling is inferior to Childs et. al's algorithm thus merely using PEA to approximate the selective transformations does not help. We also need to use FPQS to improve our approximation as described in Sec. 3.2.

Using Eq. (46), we find that our algorithm is successful as long as $(qn)^{q+1} x \ln x \ll 1$ which is satisfied as long as $q = O(\ln(x \ln x)) \approx O(\ln x)$ ignoring $O(\ln \ln x)$ term. Then Eqs. (41) and (59) imply that the number of applications of U_s required by PEA while using the improved approximation scheme is determined by $2^{l+1}gt \gg q$ or

$$2^l = O\left(\frac{q}{gt}\right) = O\left(\frac{\ln x}{gt}\right) = O\left(\frac{\Gamma \ln x}{g}\right) \quad (67)$$

where we have used Eq. (47).

The running time of our algorithm is measured in terms of number of applications of U_s used by our algorithm. This is a product of three terms: 1. $O(x \ln x)$, which is the required number of selective transformations, 2. $O(q^2) = O(\ln^2 x)$, which is the number of runs of PEA used by the FPQS-based scheme to approximate each selective transformation and 3. $O(\Gamma \ln x / g)$, which is the number of applications of U_s required for each run of PEA. Putting $x = \Gamma/g$, we find that the running time is

$$T = O\left(\frac{\Gamma^2}{g^2} \ln^4 \frac{\Gamma}{g}\right). \quad (68)$$

Thus we recover the inverse-square time scaling behaviour of quantum adiabatic evolution. Though our scaling is still inferior to that of quantum adiabatic evolution by power of logarithmic factors, we note that scaling with respect to power of g is same and this is better than the inverse-cubic dependence of Childs et.al's algorithm. We also note that apart from the selective transformations, our algorithm also requires M projective measurements on the ground states of intermediate Hamiltonians. Such measurements can also be approximated using PEA and we can show in the same way as shown in the appendix of Childs et. al's paper [1] that these approximate measurements are enough for the success of our algorithm.

V. CONCLUSION

In this paper, we have presented important applications of the concept of fixed-point quantum search algorithm which achieves a monotonic convergence towards the desired final state. We note that the advantage in our algorithm is possible only because we were able to increase the success probability of getting successive ground states using FPQS from $(1 - (\Gamma/g)^2)$ to $(1 - (\Gamma/g)^q)$ for $q > 2$. Normally, when we have two mutually close quantum states $|\alpha\rangle$ and $|\beta\rangle$ then a measurement of $|\alpha\rangle$ yields $|\beta\rangle$ with the probability $|\langle \alpha | \beta \rangle|^2 = 1 - \epsilon$ where $\epsilon \ll 1$. This process can be repeated to boost the

success probability to $1 - \epsilon^n$ provided we can always prepare the state $|\alpha\rangle$, which is possible only if we know this state. Since we don't know the ground states of intermediate Hamiltonians H_s , we cannot use such procedure in the case of Childs et. al's algorithm. FPQS offers a similar procedure but relaxes the criteria of knowing the state $|\alpha\rangle$ and uses only an oracular access to $|\alpha\rangle$ in order to apply a selective-phase rotation. Such an oracular access

is approximated by the phase-estimation algorithm and the approximation is significantly improved further using FPQS. This property of FPQS to give a performance similar to repeated measurements without a knowledge of the initial state is something that we have used in designing our algorithm. We hope that FPQS can also be used to advantage in other contexts.

-
- [1] A. Childs et al., Phys. Rev. A **66**, 032314 (2002).
 - [2] L.K. Grover, Phys. Rev. Lett. **95**, 150501 (2005).
 - [3] T. Tuli, L.K. Grover, and A. Patel, Quant. Inform. and Comput. **6**, 483 (2006).
 - [4] A. Messiah, Quantum Mechanics, Chapter XVII (Dover, New York, 1999).
 - [5] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, arXiv.org:quant-ph/0001106.
 - [6] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).
 - [7] L.K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
 - [8] S. Chakraborty, J. Radhakrishnan, and N. Raghunathan, Proc. APPROX-RANDOM 2005, LNCS 3624 (Springer, Berlin, 2005), p. 245.